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An upper bound on structurally stable linear regulation of a parameter-dependent family of control systems

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Abstract: We investigate the use of a linear compensator to regulate a parameter-dependent control system. This system may be linear or nonlinear. Our goal is to design a compensator that will ensure stable regulation over a wide range of parameter values. This paper reports a fundamental limitation on achieving this aim. In particular, we show that systems with a special, unregulatable, structure form hypersurfaces on the open-loop equilibrium manifold. Such systems include, but are not restricted to, those with transmission zeros at the origin. These surfaces partition the open-loop equilibrium manifold into disjoint open sets. We show that a linear compensator designed to regulate a system in some such partition must fail to regulate almost all systems in an adjacent partition. Therefore, by consideration of the open-loop system only, we derive upper bounds on the robustness of any linear regulator. We discuss some other qualitative aspects of the closed-loop dynamics.

Keywords: Linear control; regulation; robustness.

1. Introduction

Our objective is asymptotic regulation of the output vector (assumed to also be the measurement vector) of a parameter-dependent control system at an equilibrium point. All results apply to parameter-dependent linear or nonlinear systems. We consider the situation that the parameter values, while constant, are uncertain. As the parameter vector varies, the equilibrium point may also vary. We wish to select a linear compensator that will ensure regulation over a wide range of parameter values. Many authors have considered the use of linear theory to regulate nonlinear systems, e.g. [6, 4]. Our focus lies on the lines of Francis and Wonham [6], who show that a structurally stable regulator designed for a linear approximation of an analytic system will regulate the nonlinear system in some neighborhood of the equilibrium point, but we are concerned with behavior in the large, rather than locally. Most previous studies do not address parameter dependence, rather they require that the initial conditions be sufficiently close to equilibrium. Kwatny, et al. [9] address parameter dependence directly. They show that a structurally stable linear regulator designed for a linear approximation to the nonlinear system according to the standard theory [6, 5, 10] that asymptotically rejects constant disturbances will regulate the nonlinear system in some neighborhood of the design point. In this case the neighborhood includes small variations in the parameter vector, and so considers small variations in the location of the equilibrium point as well as the initial conditions. Theorem 3.2 in [9] states that certain structural properties of the open-loop nonlinear system determine an upper bound on the allowable variation in the parameter vector. This note expands on that result.

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Our approach is as follows. We show that the set of open-loop regulated equilibrium points forms a regular submanifold of the state-control-parameter product space, which we call the equilibrium manifold. The equilibrium manifold is naturally partitioned by codimension-one submanifolds of 'singular points'. These singular points are distinguished by the property that the system pencil of the linearized open-loop dynamics is singular. This may be due to a transmission zero at the origin, but other causes exist. We select a design point in any such partition, which we call the design partition. Any linear compensator ensuring structurally stable regulation at the design point must contain an observable and controllable internal model of the disturbance [6]. We use these properties of the compensator to show that the closed-loop equilibrium manifold is diffeomorphic to the open-loop equilibrium manifold, and that the submanifold of open-loop singular points is diffeomorphic to a submanifold of closed-loop singular points. Therefore, the closed-loop equilibrium manifold is also partitioned. The closed-loop singular points correspond to systems whose linearized dynamics have an eigenvalue at the origin. By genericity arguments, if there is a stable closed-loop equilibrium point in one partition then almost every closed-loop equilibrium point in any adjacent partition is unstable. But the partition diffeomorphic to the design partition contains such a point, and its boundaries correspond to the boundaries of the design partition. Therefore, the design partition represents an upper bound on the parameter robustness of any linear compensator.

We make repeated use of the following useful and well-known theorem.

Theorem 0. *Given C^∞ manifold $\mathcal{N}$ of dimension n and C^∞ manifold $\mathcal{M}$ of dimension m, and a C^∞ mapping $F:\mathcal{N}\to\mathcal{M}$, if F is of constant rank k on $\mathcal{N}$, then for any $q\in F(\mathcal{N})$, $F^{-1}(q)$ is a closed, regular, submanifold of $\mathcal{N}$ with dimension $n-k$.*

Proof. See [3, pp. 79–80]. □

2. The open-loop equilibrium manifold

We consider a regulated dynamic system defined by the differential and algebraic equations:

$$\dot{x} = f(x, u, c),$$

$$y = g(x, c).$$

Here $x\in\mathcal{X}\subset\mathbb{R}^n$ is a vector of states, $u\in\mathcal{U}\subset\mathbb{R}^p$ is a vector of controls, $c\in\mathcal{C}\subset\mathbb{R}^k$ is a vector of constant parameters, and $y\in\mathcal{Y}\subset\mathbb{R}^p$ is a vector of controlled outputs, all of which we assume are available for measurement. Here each $\mathbb{R}^i$ is an i-dimensional Euclidean space. We assume f and g are 'sufficiently smooth'. For convenience, we take 'sufficiently smooth' to be infinitely differentiable, written C^∞. Our control objective is to asymptotically drive y to zero, while maintaining internal stability. Note that we assume that the number of controls is equal to the number of outputs. This assumption is consistent with standard techniques, as, locally, additional controls give no added benefit. One result of this study, however, is that added controls may significantly increase robustness. We discuss this in the conclusions section.

Define the open-loop regulated equilibrium function Φ_{ol},

$$\Phi_{ol}(x, u, c) \stackrel{\text{def}}{=} \begin{bmatrix} f(x, u, c) \\ g(x, c) \end{bmatrix}.$$

A point (x, u, c) satisfying the open-loop regulated equilibrium condition, $\Phi_{ol}(x, u, c) = 0$, is an *open-loop regulated equilibrium point*. The set of all such points is denoted by $\mathcal{E}_{ol}$.

We consider Φ_{ol} as a C^∞ mapping from $\mathscr{X} \times \mathscr{U} \times \mathscr{C}$ to $\mathbb{R}^{n+p}$. Then the rank of Φ_{ol} (at a given point) is defined to be rank of the following matrix of partial derivatives:

$$\operatorname{rank} \Phi_{ol} = \operatorname{rank} \begin{bmatrix} \frac{\partial f}{\partial x}(x, u, c) & \frac{\partial f}{\partial u}(x, u, c) & \frac{\partial f}{\partial c}(x, u, c) \\ \frac{\partial g}{\partial x}(x, c) & 0 & \frac{\partial g}{\partial c}(x, c) \end{bmatrix}.$$

Consider $\mathscr{X} \times \mathscr{U} \times \mathscr{C}$ as a $(n + p + k)$-dimensional manifold with the product topology. We *assume* there exists $\mathscr{D}$, an open submanifold of $\mathscr{X} \times \mathscr{U} \times \mathscr{C}$, such that

(WRC) $\operatorname{rank} \Phi_{ol} = n + p$

for all (x, u, c) on $\mathscr{D}$. We call this the *weak regularity* condition. Therefore, by Theorem 0, $\mathscr{E}_{ol} = \Phi^{-1}(0)$ is a closed, regular, submanifold of $\mathscr{D}$ with dimension $(n + p + k) - (n + p) = k$. We refer to $\mathscr{E}_{ol}$ as the *open-loop equilibrium submanifold.* $\mathscr{E}_{ol}$ has the relative topology, i.e. $\mathscr{U} \subset \mathscr{E}_{ol}$ is open if and only if $\mathscr{U} = \mathscr{V} \cap \mathscr{E}_{ol}$, with $\mathscr{V}$ open in $\mathscr{D}$.

We further distinguish certain points of $\mathscr{E}_{ol}$. If the matrix

$$\Psi_{ol}(x, u, c) = \begin{bmatrix} \frac{\partial f}{\partial x}(x, u, c) & \frac{\partial f}{\partial u}(x, u, c) \\ \frac{\partial g}{\partial x}(x, c) & 0 \end{bmatrix}$$

satisfies the condition

(SRC) $\operatorname{rank} \Psi_{ol}(x, u, c) = n + p$

at a point $(x, u, c) \in \mathscr{E}_{ol}$, we say that (x, u, c) satisfies the *strong regularity* condition, and we call (x, u, c) a *regular point*. A point at which the strong regularity condition fails to hold is called a *singular point*. This nomenclature requires some justification, for which we must make one more assumption.

Consider the C^∞ function $D: \mathscr{E}_{ol} \to \mathbb{R}$ defined by $D(x, u, c) = \det \Psi_{ol}(x, u, c)$. We assume that rank $D = 1$ on $\mathscr{E}_{ol}$. We loosely claim that this assumption is not restrictive, but have not pursued it further. For now we note that it allows us to apply Theorem 0 to the set of points $\mathfrak{d} = D^{-1}(0)$. Then by the Theorem 0, $\mathfrak{d}$ is a closed, regular, submanifold of $\mathscr{E}_{ol}$ with dimension $k - 1$. But $\mathfrak{d}$ is exactly the set of singular points. So we see that the singular points form a submanifold of codimension one, and are in that sense 'singular', while the regular points form open submanifolds of dimension k. Since $\mathfrak{d}$ is of constant dimension, and closed, it cannot have an edge in $\mathscr{E}_{ol}$. Therefore, $\mathfrak{d}$ must partition $\mathscr{E}_{ol}$ into open disjoint submanifolds of regular points. We call these submanifolds *sheets*. We refer to $\mathfrak{d}$ as the *sheet boundary*. As a consequence of these definitions and assumptions, a sheet contains no singular points.

Note that a point is singular if the linearized system at that point has a transmission zero at the origin. This is not the only way a point may be singular, though. In terms of the Kronecker invariants of the system pencil of the linearized system, the point will also be singular if the corresponding linearization has row or column Kronecker indices. In the standard terminology [7] *if, at a point* $(x^0, u^0, c^0) \in \mathscr{E}_{ol}$, *the corresponding system pencil is singular then the point is singular.*

3. Linear regulation

We linearize the system equations about an equilibrium point, $(x^0, u^0, c^0) \in \mathscr{E}_{ol}$:

$$\dot{\delta x} = A^0 \delta x + B^0 \delta u + E^0 \delta c,$$

$$\delta y = C^0 \delta x + F^0 \delta c,$$

where A^0, B^0, C^0, E^0, and F^0 are the partials evaluated at (x^0, u^0, c^0).

We want to design a compensator that will regulate the measurement vector δy in a neighborhood of (x^0, u^0, c^0). For a sufficiently small neighborhood of c^0 this system looks like a linear system with a small constant disturbance. We apply the standard linear theory of structurally stable regulation [6, 5, 10] theory to the linearized system. We assume the following necessary conditions:

(LR1) (A^0, B^0) is stabilizable,

(LR2) (C^0, A^0) is detectable,

(LR3) $\operatorname{rank}\begin{bmatrix} A^0 & B^0 \\ -C^0 & 0 \end{bmatrix} = n + p.$

(LR3) is the requirement that the open-loop system have no transmission zeros at the poles of the disturbance, in this case a constant. We recognize that (LR3) is equivalent to (SRC), i.e. we require the design point (x^0, u^0, c^0) to be a regular point. Strong regularity implies that C^0 is full row rank, and that the number of controls is greater than or equal to the number of outputs. As already mentioned, we assume these to be equal.

A general linear compensator has form

(GLCa) $\dot{z} = Gz + Hy,$

(GLCb) $\delta u = Kz + Ly.$

Any compensator ensuring structurally stable regulation must contain a (at least) p-fold internal model of the (constant) disturbance, observable from u and controllable form y [6]. This induces a decomposition of the compensator state space

$$\mathscr{X} = \mathscr{X}_1 \oplus \mathscr{X}_2$$

where $\mathscr{X}_1$ is G-invariant and $\mathscr{X}_2$ corresponds to the internal model. These necessary properties are equivalent to the following:

(C1) In some basis G has the structure

$$G = \begin{bmatrix} G_{11} & G_{12} \\ 0^{(r\times q)} & 0^{(r\times r)} \end{bmatrix},$$

where G_{11} is a $q\times q$ matrix and G_{12} is a $q\times r$ matrix with $r \geq p$. The other matrices are partitioned accordingly as

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},$$

$$K = [K_1 \quad K_2]$$

and L is unchanged.

For the internal model to be observable from u, we must have

$$\ker\begin{bmatrix} G_{11} & G_{12} \\ 0^{(r\times q)} & 0^{(r\times r)} \\ K_1 & K_2 \end{bmatrix} = \{0\},$$

i.e.

$$\ker\begin{bmatrix} G_{11} & G_{12} \\ K_1 & K_2 \end{bmatrix} = \{0\}.$$

Since, by assumption, the number of controls equals the number of outputs p, this matrix has $q+r$ columns and $q+p$ rows. Thus, a necessary condition for this matrix to have full column rank is $r \le p$. We conclude that $r = p$, and that the internal model includes exactly p duplications of the disturbance. We summarize as follows.

(C2) G and K satisfy

$$\operatorname{rank}\begin{bmatrix} G_{11} & G_{12} \\ K_1 & K_2 \end{bmatrix} = q + p,$$

i.e.

$$\begin{bmatrix} G_{11} & G_{12} \\ K_1 & K_2 \end{bmatrix}$$

is invertible.

(C3) H_2 is a $p \times p$ invertible matrix.

4. The closed-loop equilibrium submanifold

Given a particular compensator (GLC) satisfying (C1)–(C3) define the set of equilibrium points, regular points, and singular points just as for the open-loop system. The *closed-loop regulated equilibrium function* Φ_{cl} is

$$\Phi_{\text{cl}}(x, z, c) \overset{\text{def}}{=} \begin{bmatrix} f(x, u(x, z, c), c) \\ Gz + Hg(x, c) \\ g(x, c) \end{bmatrix},$$

where $u(x, z, c) = u^0 + Kz + Lg(x, c)$. Any point satisfying $\Phi_{\text{cl}}(x, z, c) = 0$ is called a *closed-loop regulated equilibrium point*. The set of all such points is denoted $\mathscr{E}_{\text{cl}}$. We characterize these points with a lemma.

4.1. Lemma *The following conditions are necessary and sufficient for* (x^*, z^*, c^*) *to be a closed-loop regulated equilibrium point*:

(CL1) *The point* $(x^*, u^0 + Kz^*, c^*)$ *is an open-loop regulated equilibrium point.*

(CL2) $Gz^* = 0$.

Denote by $\mathscr{D}_{\text{cl}}$ the subset of $\mathscr{X} \times \mathscr{Z} \times \mathscr{C}$ defined by

$$\mathscr{D}_{\text{cl}} = \{(x, z, c)\colon (x, u^0 + Kz + Lg(x, c), c) \in \mathscr{D}\}.$$

Φ_{cl} is a C^∞ mapping from $\mathscr{D}_{\text{cl}}$ to $\mathbb{R}^{n+q+2p}$. We can relate the rank of Φ_{cl} to the rank of Φ_{ol}. The rank of Φ_{cl} is by definition the rank of the matrix

$$\operatorname{rank} \Phi_{\text{cl}} = \operatorname{rank}\begin{bmatrix} \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial u} L \dfrac{\partial g}{\partial x} & \dfrac{\partial f}{\partial u} K & \dfrac{\partial f}{\partial c} + \dfrac{\partial f}{\partial u} L \dfrac{\partial g}{\partial c} \\ H \dfrac{\partial g}{\partial x} & G & H \dfrac{\partial g}{\partial c} \\ \dfrac{\partial g}{\partial x} & 0 & \dfrac{\partial g}{\partial c} \end{bmatrix}.$$

Apply elementary row and column operations. Then

$$\operatorname{rank}\Phi_{\mathrm{cl}} = \operatorname{rank}\begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial u}K & \dfrac{\partial f}{\partial c} \\ 0 & G & 0 \\ \dfrac{\partial g}{\partial x} & 0 & \dfrac{\partial g}{\partial c} \end{bmatrix}.$$

Expand G and K,

$$\operatorname{rank}\Phi_{\mathrm{cl}} = \operatorname{rank}\begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial u}K_1 & \dfrac{\partial f}{\partial u}K_2 & \dfrac{\partial f}{\partial c} \\ 0 & G_{11} & G_{12} & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{\partial g}{\partial x} & 0 & 0 & \dfrac{\partial g}{\partial c} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial u}K_1 & \dfrac{\partial f}{\partial u}K_2 & \dfrac{\partial f}{\partial c} \\ \dfrac{\partial g}{\partial x} & 0 & 0 & \dfrac{\partial g}{\partial c} \\ 0 & G_{11} & G_{12} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial c} & 0 \\ \dfrac{\partial g}{\partial x} & 0 & \dfrac{\partial g}{\partial c} & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & K_1 & K_2 & 0 \\ 0 & 0 & 0 & I \\ 0 & G_{11} & G_{12} & 0 \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial c} & 0 \\ \dfrac{\partial g}{\partial x} & 0 & \dfrac{\partial g}{\partial c} & 0 \\ 0 & 0 & 0 & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & K_1 & K_2 & 0 \\ 0 & 0 & 0 & I \\ 0 & G_{11} & G_{12} & 0 \end{bmatrix}$$

So, by (WRC) and (C3), $\operatorname{rank}\Phi_{\mathrm{cl}} = n + p + q$ everywhere on $\mathscr{D}_{\mathrm{cl}}$. Apply Theorem 0 to give the following lemma.

Lemma 4.2. *The set of points $\mathscr{E}_{\mathrm{cl}} = \Phi_{\mathrm{cl}}^{-1}(0)$ is a closed, regular, submanifold of $\mathscr{D}_{\mathrm{cl}}$ with dimension k.*

We say that Φ_{cl} satisfies the *closed-loop weak regularity condition* on $\mathscr{D}_{\mathrm{cl}}$. In fact, we can show much more than this. Lemma 4.1, and the compensator properties (C1)–(C3) give the following theorem.

Theorem 4.3. *The open-loop and closed-loop equilibrium manifolds are diffeomorphic.*

Proof. Let (x^*, z^*, c^*) be any closed-loop regulated equilibrium point. We can define a C^∞ map $\mathfrak{D}: \mathscr{E}_{\mathrm{cl}} \to \mathscr{E}_{\mathrm{ol}}$ by

($\mathfrak{D}$) $(x^*, z^*, c^*) \mapsto (x^*, u^0 + Kz^*, c^*)$.

Define $u^* = u^0 + Kz^*$. Clearly, $\mathscr{D}$ is 1–1 if $z^* \mapsto u^*$ is 1–1. Let z_1^* and z_2^* be such that $u^0 + Kz_1^* = u^0 + Kz_2^*$. Then, together with (CL2), we have

$$\begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \\ K_1 & K_2 \end{bmatrix} (z_1^* - z_2^*) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and so, by (C2), $z_1^* = z_2^*$ and $\mathfrak{D}$ is 1–1. Now consider any $(x', u', c') \in \mathscr{E}_{\mathrm{ol}}$. Clearly, $\mathfrak{D}$ is onto if there exists z^* such that $u' = u^0 + Kz^*$ and (CL2) is satisfied. That is, if there exists a solution to the equation

$$\begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \\ K_1 & K_2 \end{bmatrix} z^* = \begin{bmatrix} 0 \\ 0 \\ u^0 - u' \end{bmatrix}.$$

But this is equivalent to

$$\begin{bmatrix} G_{11} & G_{12} \\ K_1 & K_2 \end{bmatrix} z^* = \begin{bmatrix} 0 \\ u^0 - u' \end{bmatrix},$$

which, by (C2), always has a solution. So $\mathfrak{D}$ is onto and, in fact, we see that

$$z^* = \begin{bmatrix} G_{11} & G_{12} \\ K_1 & K_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ u^0 - u' \end{bmatrix}$$

is the only nontrivial part of the inverse relation, so $\mathfrak{D}^{-1}$ is also C^∞. We conclude that $\mathfrak{D}$ is a diffeomorphism. □

$\mathscr{E}_{\mathrm{ol}}$ and $\mathscr{E}_{\mathrm{cl}}$ are diffeomorphic. Next consider the set of points $\mathfrak{D}(\mathfrak{b})$ – the image of the open-loop sheet boundaries under $\mathfrak{D}$. Clearly, $\mathfrak{D}(\mathfrak{b})$ is a subset of $\mathscr{E}_{\mathrm{cl}}$. The diffeomorphism must preserve the differential and topological properties of $\mathfrak{b}$. So $\mathfrak{D}(\mathfrak{b})$ is a closed regular submanifold (of $\mathscr{E}_{\mathrm{cl}}$) of codimension one and partitions $\mathscr{E}_{\mathrm{cl}}$ into disjoint open sets. We can call these *sheets* of the closed-loop equilibrium manifold and $\mathfrak{D}(\mathfrak{b})$ the *closed-loop sheet boundary.* We now show the following result on stability.

Lemma 4.4. *The linearized closed-loop dynamics have an eigenvalue at the origin if and only if the corresponding open-loop equilibrium point is singular, i.e. if and only if the closed-loop system is on the closed-loop sheet boundary.*

Proof. The dynamics of the closed-loop system are determined by the eigenvalues of the following Jacobian matrix:

$$J_{\mathrm{cl}} = \begin{bmatrix} \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial u} L \dfrac{\partial g}{\partial x} & \dfrac{\partial f}{\partial u} K \\ H \dfrac{\partial g}{\partial x} & G \end{bmatrix}.$$

Expand G, H, and K:

$$J_{\mathrm{cl}} = \begin{bmatrix} \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial u} L \dfrac{\partial g}{\partial x} & \dfrac{\partial f}{\partial u} K_1 & \dfrac{\partial f}{\partial u} K_2 \\ H_1 \dfrac{\partial g}{\partial x} & G_{11} & G_{11} \\ H_2 \dfrac{\partial g}{\partial x} & 0 & 0 \end{bmatrix}.$$

By (C3), H_2 is invertible, so

$$\operatorname{rank} J_{\mathrm{cl}} = \operatorname{rank}\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial u}K_1 & \frac{\partial f}{\partial u}K_2 \\ 0 & G_{11} & G_{12} \\ \frac{\partial g}{\partial x} & 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial u}K_1 & \frac{\partial f}{\partial u}K_2 \\ \frac{\partial g}{\partial x} & 0 & 0 \\ 0 & G_{11} & G_{12} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial u} & 0 \\ \frac{\partial g}{\partial x} & 0 & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & K_1 & K_2 \\ 0 & G_{11} & G_{12} \end{bmatrix}.$$

Thus, we conclude that the closed-loop linearized dynamics have an eigenvalue at the origin if and only if the closed-loop regulated equilibrium point is diffeomorphic to an open-loop singular point. □

We also call $\mathfrak{D}(\mathfrak{d})$ the submanifold of *static singular points*. We see that the closed-loop sheets can contain no static singular points.

We want to use these results to examine the properties of closed-loop equilibrium points in adjacent sheets. Intuitively, we expect that as a stable system 'crosses' the sheet boundary a closed-loop eigenvalue 'crosses' into the right half-plane. However, we think of systems from a parameter-dependent family as being constant once selected, so we must be careful about 'moving' systems around in parameter space. Also we will have to make some indirect arguments about the dynamics away from the singular point, since by using elementary row and column operations instead of similarity transformations, we destroyed the more general eigenstructure.

5. One-parameter families

Given any two points in a sheet then any smooth curve connecting them, lying completely within the sheet, contains no singular points. Similarly, given any two points in adjacent sheets there must exist a smooth curve connecting the points that contains one and only one singular point – from the sheet boundary. Furthermore, every smooth curve connecting the two points must contain at least one point from the sheet boundary.

Consider the generic behavior of the eigenvalues of the linearized dynamics of a one-parameter family of dynamic systems. Because the entries of the Jacobian matrix depend smoothly on the parameter, and because the eigenvalues of a matrix depend smoothly on its entries, the eigenvalues depend smoothly on the parameter. So we have $2n + p$ functions, $\lambda(x, z, c)$, from $\mathscr{D}_{\mathrm{cl}}$ to $\mathbb{C}$. Typically, we would not expect any two of these functions to have a simultaneous zero. Because the dynamics represent a physical system, nonreal eigenvalues must occur as complex-conjugate pairs. However, we would typically expect no two complex-conjugate pairs to have zero real part simultaneously, nor would we expect a complex-conjugate pair to have zero real part simultaneously with a zero of a real eigenvalue. All we need for the following is that the double-zero eigenvalue is not generic in one-parameter families of dynamic systems (see e.g. [1] or [8]). Then we can make the following statement:

Consider a family of dynamic systems dependent on a single parameter, v. The number of poles of the linearized dynamics in the right half-plane generically change only in one of two ways. Either a single eigenvalue intersects the origin or a single nonzero complex-conjugate pair of eigenvalues intersects the imaginary axis.

We can always redefine the parameter so that the critical case corresponds to $v = 0$. Generically, the intersection must be transverse, that is, the eigenvalues must actually cross from one half-plane to the other as the parameter passes through zero. We refer to the first type as a real nonhyperbolic point, and the second type as a complex nonhyperbolic point.

In general, the multiple parameter nonlinear case will be much more complicated. The following theorem, our main results, requires only the one-parameter case.

Theorem 5.1. *If some point in a sheet is a stable closed-loop equilibrium point then, for generic systems, every point in any adjacent sheet is an unstable closed-loop equilibrium point.*

Proof. Take the stable point and connect it to a point in any adjacent sheet by a smooth curve containing one and only one singular point. The curve can be considered a C^∞ function, $v \mapsto (x(v), u(v), c(v))$. The singular point must lie on the sheet boundary. Let the zero value of the parameter v correspond to the singular point and let negative (positive) values of v correspond to points in the first (second) sheet. Generically, a single real pole crosses through the origin when $v = 0$. By the previous results we know that the linearized closed-loop dynamics have no zero eigenvalues for $v < 0$. Along the curve, some poles may have crossed the imaginary axis while $v < 0$, but only as complex-conjugate pairs. Thus, the number of poles in the right half-plane must be zero or even. The zero eigenvalue at $v = 0$ may be crossing from right to left or from left to right. In either case, for $v > 0$ there is an odd number of poles in the right half-plane, proving the theorem. □

These arguments also give the following corollary.

Corollary 5.2. *Along any smooth curve within a sheet, a system can lose stability only by one or more nonzero, complex-conjugate pair of poles of the linearized closed-loop dynamics crossing through the imaginary axis.*

Theorem 5.1 and Corollary 5.2 refer to the behavior of the linearized dynamics. We are interested in the behavior of the corresponding nonlinear system. In general, there is little we can say for sure, but if we again consider the generic one-parameter case we have the following results. With only a single real pole, or a single nonzero pair of poles, crossing the imaginary axis we can restrict our focus to behavior on a planar invariant manifold. When the equilibrium point of the linearized dynamics is a real nonhyperbolic point a one-parameter family of planar nonlinear systems generically experiences a saddle-node bifurcation. When the equilibrium point is a complex nonhyperbolic point the one-parameter family of planar nonlinear systems generically undergoes a Hopf bifurcation. (See again [1] or [8]). The remaining dynamics are hyperbolic and so structurally stable. Therefore, we have the following corollaries for nonlinear systems.

Corollary 5.3. *Along any smooth curve within a sheet, a generic system will lose stability only by a Hopf bifurcation.*

Corollary 5.4. *If a generic system is stable everywhere on a sheet then it loses stability at the sheet boundary and the mechanism is a saddle-node bifurcation.*

In all the above results the use of one-parameter families allows us to apply standard results on generic bifurcations. If we try to consider the general multiparameter case the number of generic possibilities quickly becomes large.

5.1. A remark on the results

The above analysis assumed a structurally stable linear regulator. We consider this assumption unrestrictive because structural stability is a natural requirement at the design point to ensure robustness to small nonlinearities and unmodelled uncertainties, i.e. everything other than the explicit parameter dependence. However, we would have preferred to prove an upper bound on all regulators, and then obtained the result for structurally stable compensators as a special case. We are still pursuing the more general case. It is worthwhile to note how the preceding argument changes and where it breaks down.

Regulation requires an internal model, but dropping the structural stability requirement means that the model need not be a p-fold duplicate. Then condition (C1) becomes

(C1′) In some basis, G has the structure

$$G = \begin{bmatrix} G_{11} & G_{12} \\ 0^{(r\times q)} & 0^{(r\times r)} \end{bmatrix},$$

where previously we found that $r = p$, but now we have $0 < r \leq p$. We proceed as before to determine that

$$\operatorname{rank} \Phi_{\mathrm{cl}} = \operatorname{rank} \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial c} & 0 \\ \dfrac{\partial g}{\partial x} & 0 & \dfrac{\partial g}{\partial c} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & K_1 & K_2 & 0 \\ 0 & 0 & 0 & I \\ 0 & G_{11} & G_{12} & 0 \end{bmatrix}.$$

The second matrix must still have full column rank, but need not be square, and so need not have full row rank. So the equality rank $\Phi_{\mathrm{cl}} = n + p + q$ becomes rank $\Phi_{\mathrm{cl}} \leq n + p + q$. If the rank is constant, which is not guaranteed, we have that the set of closed-loop equilibrium points is a closed regular submanifold of dimension greater than or equal to that of the open-loop equilibrium manifold. So these manifolds will not generally be diffeomorphic, and the structure of our argument breaks down.

6. Conclusions

In this note we studied the use of a single linear compensator to regulate a parameter-dependent family of control systems. We found that the structure of the open-loop system places a fundamental limitation on the robustness of any such compensator.

In particular, we considered the product space of states, controls, and parameters as a C^∞ manifold. Under certain regularity conditions the set of all open-loop equilibrium points forms a closed, regular, submanifold (the 'open-loop equilibrium manifold') with dimension equal to the number of parameters. Then, under some additional rank assumptions, the set of all 'singular' points – points at which the system pencil of the linearized system loses rank due to a zero at the origin or some other degeneracy – forms a closed, regular submanifold of the equilibrium manifold, with codimension one. This submanifold partitions the open-loop equilibrium manifold into disjoint open sets ('sheets').

We show that selecting a linear compensator at some regular point, i.e. at a point within a sheet, defines a diffeomorphism between the open-loop and closed-loop equilibrium manifolds. Recall that such a compensator is guaranteed to regulate all systems in a neighborhood of the design point, and that no such regulator can be designed at a singular point. Of course, the diffeomorphism preserves the partitioning, and on the closed-loop equilibrium manifold the sheet boundaries represent the set of points at which the linearized closed-loop dynamics have a pole at the origin. We then argue that generically the closed-loop system, if it has not done so already, must lose stability across a sheet boundary. Thus, the sheet structure, which is a property of the open-loop system and independent of the compensation method, represents an upper bound on the allowable parameter values for any single linear compensator.

Another genericity argument relates the behavior of the linearized system to that of the original nonlinear system. We contend that the original nonlinear system must either lose stability within the sheet via a Hopf bifurcation or at the sheet boundary via a saddle-node bifurcation. Because we never specify the design point, or the details of the linear compensator, the results shown here apply to linear regulation of almost all smooth control systems.

The structure of the open-loop equilibrium manifold and an example showing the consequences for regulation are explored in some detail in [2].

We made several assumptions. We assumed that the control space had the same dimension as the output space. This assumption does not affect the design of a compensator in a small neighborhood of the design point, but it is easy to see that additional controls may change the global behavior. Consider, for example, a situation in which two controls become linearly dependent. This will cause the system pencil to lose rank, and so restrict the robustness of any linear compensator. Adding an additional independent control would eliminate this situation. We also assumed that the rank of $D(x, u, c) = \det \Psi_{ol}(x, u, c)$ was one everywhere on the open-loop equilibrium manifold.

The theorems derived involved only points on the equilibrium manifolds. Therefore, we could interpret our final result as saying that any stabilizing compensator enforcing regulation must fail in an adjacent sheet. A stabilizing compensator that relaxes the regulation conditions slightly (an almost-regulating compensator) might be able to maintain stability over a much wider range.

References

[1] D.K. Arrowsmith and C.M. Place, *An Introduction to Dynamical Systems* (Cambridge Univ. Press, New York, 1990).
[2] J. Berg and H. Kwatny, Unfolding the singular system pencil, *Proc. IFAC 13th World Congress.* Sydney, Australia, 1992.
[3] W.M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry* (Academic Press, San Diego, 1986).
[4] C.A. Desoer and C.-A. Lin, Tracking and disturbance rejection of MIMO nonlinear systems with PI controller, *IEEE Trans. Automat. Control* **AC-30** (1985) 861–867.
[5] B.A. Francis, The linear multivariable regulator problem, *SIAM J. Control Optim.* **15** (1977) 486–505.
[6] B.A. Francis and W.M. Wonham, The internal model principle of control theory, *Automatica* **12** (1976) 457–465.
[7] F.R. Gantmacher, *The Theory of Matrices, Vol. 2* (Chelsea, New York, 1959).
[8] J. Hale and H. Koçak, *Dynamics and Bifurcations*, (Springer, New York, 1991).
[9] H.G. Kwatny, W.H. Bennett and J.M. Berg, Regulation of relaxed stability aircraft, *IEEE Trans. Automat. Control* **AC-36** (1990) 1323–1325.
[10] H.G. Kwatny and K.C. Kalnitsky, On alternative methodologies for the design of robust linear multivariable regulators, *IEEE Trans. Automat. Control* **AC-23** (1978) 930–933.